International Journal of Theoretical & Applied Sciences, **8**(1): 67-71(2016)

ISSN No. (Print): 0975-1718
ISSN No. (Online): 2249-3247

A Study on a Common Fixed Point Theorem

S. Vijaya Lakshmi
Department of Mathematics,
University College of Science, Osmania University, Hyderabad, Telangana State, India.

(Corresponding author: S. Vijaya Lakshmi)
(Received 14 April, 2016 Accepted 19 June, 2016)
(Published by Research Trend, Website: www.researchtrend.net)

ABSTRACT: The aim of this paper is to prove a common fixed point theorem which generalizes the result of A. Djoudi by weaker conditions such as compatible mappings of type(P) and iterated sequence. We constructed one example in which the mappings are only compatible mappings of type(P) but not any one of compatible, compatible mappings of type (A), compatible mappings of type(B). In this paper, a method of compatible mappings of type (P) is applied to generate a common fixed point theorem on four self maps.

Keywords: Fixed point, self- maps, compatible mappings of type (P) and iterated sequence.

2010 AMS (Mathematics Classification):54H25.

1. INTRODUCTION

Two self maps S and T of a metric space (X,d) are said to be commute if ST=TS.
According to G. Jungck [1], Two self maps S and T of a metric space (X,d) are said to be compatible mappings if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, whenever $<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.
From G. Jungck and others [2],[3],[4], Two self maps S and T of a metric space (X,d) are said to be compatible mappings of type(A) if $\lim_{n\to\infty} d(STx_n, TTx_n) = 0$ and $\lim_{n\to\infty} d(TSx_n, SSx_n) = 0$ whenever $<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.
By H.K. Pathak and others [5],[6], Two self maps S and T of a metric space (X,d) are said to be compatible mappings of type(P) if $\lim_{n\to\infty} d(SSx_n, TTx_n) = 0$, whenever $<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.
In view of H.K. Pathak and others [5],[6],[9],[12], Two self maps S and T of a metric space (X,d) are said to be weak compatible mappings of type(A) if $\lim_{n\to\infty} d(STx_n, TTx_n) \le \lim_{n\to\infty} d(TSx_n, TTx_n)$ and $\lim_{n\to\infty} d(TSx_n, SSx_n) \le \lim_{n\to\infty} d(STx_n, SSx_n)$ whenever $<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.
According to H.K. Pathak and M.S. Khan [6], Two self maps S and T of a metric space (X,d) are said to be compatible mappings of type(B) ,if $\lim_{n\to\infty} d(STx_n, TTx_n) \le \frac{1}{2}\left[\lim_{n\to\infty} d(STx_n, St) + \lim_{n\to\infty} d(St, SSx_n)\right]$ and $\lim_{n\to\infty} d(TSx_n, SSx_n) \le \frac{1}{2}\left[\lim_{n\to\infty} d(TSx_n, Tt) + \lim_{n\to\infty} d(Tt, TTx_n)\right]$, whenever $<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Definition: A function $\phi:[0,\infty)\to[0,\infty)$ is said to be a contractive modulus[9],[10],[11],[15],[16], if $\phi(0)=0$ and $\phi(t)<t$ for $t>0$ For example, $\phi:[0,\infty)\to[0,\infty)$ defined by $\phi(t)=ct$, where $0\le c<1$ is a contractive modulus.

Definition: A real valued function ϕ defined on $X\subseteq R$ is said to be Upper semi continuous[12],[13],[14],[17],[18], if $\lim_{n\to\infty}\phi(t_n)\le\phi(t)$, for every sequence $<t_n>$ with $t_n\to t$ as $n\to\infty$. Every continuous function is upper semi continuous but not conversely.

II. A COMMON FIXED POINT THEOREM

Let R_+ be the set of non negative real numbers and let $\phi: R_+^5\to R_+$ be a function satisfying the following conditions:

φ is upper semi continuous in each coordinate variable and non decreasing.

$\phi(t) = \max\{\varphi(0,t,0,0,t), \varphi(t,0,0,t,t), \varphi(t,t,t,2t,0), \varphi(0,0,t,t,0)\} < t$, for any $t > 0$.

The following is the theorem proved by A. Djoudi [6].

2.1 Theorem: Let I, J, S and T be mappings from a complete metric space (X,d) into itself satisfying the conditions

(2.1.1) $S(X)\subset J(X)$ and $T(X)\subset I(X)$

(2.1.2) $d(Sx,Ty) \le \max\{\varphi(d(Ix,Jy),d(Ix,Sx),d(Jy,Ty), d(Ix,Ty),d(Jy,Sx))\}$ for all $x,y\in X$.

(2.1.3) one of S, I, T and J is continuous

(2.1.4) the pairs (S,I) and (T,J) are compatible mappings of type(B).

Then S, I, T and J have a unique common fixed point z. Furthermore z is the unique common fixed point of four mappings.

2.2 Iterated Sequence[7],[14][15]: Suppose S, I, T and J are self maps of a metric space (X, d) satisfying the condition $S(X)\subseteq J(X)$ and $T(X)\subseteq I(X)$. Then for any $x_0\in X$, $Sx_0\in S(X)$ so that there is a $x_1\in X$ with $Sx_0 = Jx_1$. Now $Tx_1\in T(X)$ and hence there is $x_2\in X$ with $Tx_1 = Ix_2$. Repeating this process to each $x_0\in X$, we get a sequence $<x_n>$ in X such that $Sx_{2n} = Jx_{2n+1}$ and $Tx_{2n+1} = Ix_{2n+2}$ for $n\ge 0$. We shall call this sequence as an "*Iterated of* x_0 "relative to the four self maps S, I, T and J.

2.3 Lemma[8]: Suppose S, I, T and J are four self maps of a metric space (X,d) for which the conditions

(a) $S(X)\subseteq J(X)$ and $T(X)\subseteq I(X)$

(b) $d(Sx,Ty) \le \phi\{d(Ix,Jy), d(Ix,Sx),d(Jy,Ty), d(Ix,Ty),d(Jy,Sx)\}$

Further if (X, d) is a complete metric space then for any $x_0\in X$ and for any of its iterated sequence $<x_n>$ relative to four self maps, the sequence $Sx_0,Tx_1,Sx_2,Tx_3,......Sx_{2n},Tx_{2n+1},.......$ converges to some point $z\in X$ (1).

The converse of the lemma is not true.

That is, suppose S, I, T and J are self maps of a metric space (X,d) satisfying the conditions(a) and (b) and even for each iterated sequence $<x_n>$ of x_0, the sequence in (1) converges, the metric space (X,d) need not be complete.

2.4 Example: Let $X=\left[0,\frac{1}{2}\right)$ with $d(x,y)=|x-y|$. Define self maps S, I, T and J of X by

$$I_x=J_x=\frac{1}{2}-x \text{ if } x\in\left[0,\frac{1}{2}\right) \text{ and } S_x=T_x=\begin{cases}\frac{1}{4} & if\ x\in\left[0,\frac{1}{4}\right]\\ \frac{1}{3} & if\ x\in\left(\frac{1}{4},\frac{1}{2}\right)\end{cases}$$

Clearly $S(X)\subseteq J(X)$ and $T(X)\subseteq I(X)$. The iterated sequence $Sx_0,Tx_1,Sx_2,Tx_3,..,Sx_{2n},Tx_{2n+1}.....$, converges to the point 1/4. But X is not a complete metric space.

III. MAIN RESULT

Theorem 3.1. Let S,I,T, and J are self maps of a metric space (X, d) satisfying the conditions

(3.1.1) S(X) $\subseteq$ J(X) and T(X) $\subseteq$ I(X)

(3.1.2) d(Sx, Ty) $\le$ {ϕ(d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), d(Ix, Ty), d(Jy, Sx))}
for all x, y in X.

(3.1.3) one of S, I, T, and J is continuous

(3.1.4) (S, I) and (T, J) are compatible mappings of type-(p).

(3.1.5) the sequence $Sx_0, Tx_1, Sx_2, Tx_3, ...Sx_{2n,} Tx_{2n+1,}.......$ converges to z$\in$X.

Then S, I, T and J have a unique common fixed point in X.

Proof: From condition (3.1.5) $Sx_{2n} \to z$ and $Tx_{2n+1} \to z$ as $n \to \infty$. (1)

Suppose S is continuous then $SSx_{2n} \to Sz$, $SIx_{2n} \to Sz$ as $n \to \infty$.

Since the pair (S,I) is compatible mappings of type (P) then $\lim_{n\to\infty} d(SSx_{2n}, IIx_{2n}) = 0$. This gives $SSx_{2n} \to Az$ as $n \to \infty$.

Therefore $\lim_{n\to\infty} SSx_{2n} = \lim_{n\to\infty} IIx_{2n} = Sz$. (2)

Put $x = Ix_{2n}, y = x_{2n+1}$ in (3.1.2), we get

$$d(SIx_{2n}, Tx_{2n+1}) \le \phi\{d(IIx_{2n}, Jx_{2n+1}), d(IIx_{2n}, SIx_{2n}), d(Jx_{2n+1}, Tx_{2n+1}), d(IIx_{2n}, Tx_{2n+1}), d(Jx_{2n+1}, SIx_{2n})\}$$

letting $n \to \infty$ and using the conditions (1) and (2), we get

$$d(Sz, z) \le \phi\{d(Sz, z), d(Sz, Sz), d(z, z), d(Sz, z), d(z, Sz)\}$$

$$d(Az, z) \le \phi\{d(Sz, z), 0, 0, d(Sz, z), d(z, Sz)\}$$

$d(Az, z) \le \phi d(Sz, z) < d(Sz, z)$, a contradiction if $Sz \ne z$.

Therefore $Sz = z$.

Since $S(X) \subseteq J(X)$ implies there exists $u \in X$ such that $z = Sz = Ju$.

To prove $Tu = z$, put $x = x_{2n}, y = u$ in (3.1.2), we get

$$d(Sx_{2n}, Tu) \le \phi\{d(Ix_{2n}, Ju), d(Ix_{2n}, Sx_{2n}), d(Ju, Tu), d(Ix_{2n}, Tu), d(Ju, Sx_{2n})\}$$

letting $n \to \infty$ and using the conditions (1) and $Sz = z$, we have

$$d(z, Tu) \le \phi\{d(z, z), d(z, z), d(z, Tu), d(z, Tu), d(z, z)\}$$

$$d(z, Tu) \le \phi\{0, 0, d(z, Tu), d(z, Tu), 0\}$$

$d(z, Tu) \le \phi d(z, Tu) < d(z, Tu)$, a contradiction if $Tu \ne z$

$d(Tu, z) = 0$ or $Tu = z$.

Therefore $Ju = Tu = z$.

Since the pair (T,J) is compatible mappings of type(P), we have
$d(TTu, JJu) = 0$. This gives $d(Bz, Tz) = 0$ or $Tz = Jz$.

To prove $Tz = z$, put $x = x_{2n}, y = z$ in (3.1.2), we get

$$d(Sx_{2n}, Tz) \le \phi\{d(Ix_{2n}, Jz), d(Ix_{2n}, Sx_{2n}), d(Jz, Tz), d(Ix_{2n}, Tz), d(Jz, Sx_{2n})\}$$

letting $n \to \infty$ and using the conditions (1) and $Tz = Jz$, we have

$$d(z, Tz) \le \phi\{d(z, Tz), d(z, z), d(Tz, Tz), d(z, Tz), d(Tz, z)\}$$

$$d(z, Tz) \le \phi\{d(z, Tz), 0, 0, d(z, Tz), d(Tz, z)\}$$

$d(z, Tz) \le \phi d(z, Tz) < d(z, Tz)$, a contradiction if $Tz \ne z$

$d(Tz, z) = 0$. Therefore $Tz = z$.

Hence $Jz = Tz = z$.

Since $T(X) \subseteq I(X)$ implies there exists $v \in X$ such that $z = Tz = Iv$.

To prove $Sv = z$, put $x = v, y = z$ in (3.1.2), we get

$$d(Sv,Tz) \leq \phi\{d(Iv,Jz), d(Iv,Sv), d(Jz,Tz), d(Iv,Tz), d(Jz,Sv)\}$$

letting $n \to \infty$ and using the condition $z = Tz = Iv$,we have

$$d(Sv,z) \leq \phi\{d(z,z), d(z,Sv), d(z,z), d(z,z), d(z,Sv)\}$$

$$d(Sv,z) \leq \phi\{0, d(z,Sv), 0, 0, d(z,Sv)\}$$

$d(Sv,z) \leq \phi d(z,Sv) < d(z,Sv)$, a contradiction if $Sv \neq z$

$d(Sv,z) = 0$ or $Sv = z$.

Therefore $z = Sv = Iv$.

Since the pair (S,I) is compatible mappings of type(P), we have

$d(SSv, IIv) = 0$ .This gives $d(Sz, Iz) = 0$ or $Sz = Iz$.

To prove $Sz = z$, put $x = z, y = z$ in (3.1.2), we get

$$d(Sz,Tz) \leq \phi\{d(Iz,Jz), d(Iz,Sz), d(Jz,Tz), d(Iz,Tz), d(Jz,Sz)\}$$

$$d(Sz,z) \leq \phi\{d(Sz,Jz), 0, 0, d(Sz,Tz), d(Sz,z)\}$$

$d(Sz,z) \leq \phi d(Sz,Jz) < d(Sz,Jz)$, a contradiction if $Sz \neq z$.

$d(Sz,z) = 0$ or $Sz = z$.

Therefore $z = Sz = Iz$.

Since $Sz = Iz = Jz = Tz = z$, we get z is a common fixed point of S, I, J and T. The uniqueness of the fixed point can be easily proved.

DISCUSSION

From the example (2.4), clearly the pairs (S, I) and (T, J) are not commutative and it can be easily verified that the mappings are not compatible, compatible mappings of type(A), weak compatible of type(A) and also not compatible of type(B) but they are compatible of type(P).

CONCLUSION

In the above mentioned process a common fixed point is generated and is unique. In fact 1/4 is the unique common fixed point for the four self maps.

REFERENCES

[1]. Jungck. G, (1986). Compatible mappings and common fixed points. *Internat. J. Math. & Math. Sci.,* **9**, pp: 771-778.

[2]. R.P. Pant, (1999). A Common fixed point theorem under a new condition. *Indian J. of Pure and App. Math.*, **30**(2), pp: 147-152.

[3]. Jungck, G. (1988). Compatible mappings and common fixed points. *Internat. J. Math. & Math. Sci.,* **11**, pp: 285-288.

[4]. Jungck, G. and Rhoades. B.E., (1998). Fixed point for set valued functions without continuity. *Indian J. Pure. Appl. Math.,* **29**(3), pp: 227-238.

[5]. H.K. Pathak, (1989). Common fixed point theorem for weak commuting mappings. *Bull. Calcutta. Math. Soc*., **81:** 455-466.

[6]. H.K. Pathak, R.K. Verma, S.M. Kang and S.M. Khan (2006). Fixed points for weak compatible type and Parametrically φ(ϵ, δ;a)-contraction mappings. *Int. I. Pure Appl. Math,* **26**: 247-263.

[7]. Bijendra Singh and S. Chauhan (1998). On common fixed poins of four mappings. *Bull. Cal. Math.Soc.,* **88**, pp: 301-308.

[8]. A. Djoudi, (2003). A common fixed point theorem for compatible mappings of type (B) in complete metric spaces. *Demonstr. Math.* Vol. **XXXVI**, No. 2, 463-470.

[9]. A.S. Saluja, Mukesh Kumar Jain and Pankaj Kumar Jhade, (2002). Weak semi compatibility and fixed point theorems. *Bulletin of International mathematical Virtual Institute,***2**, 205-217.

[10]. Popa, V. (2001). A general common fixed point theorem for weakly compatible mappings in compact metric spaces, *Turk. J. Math.,* **25**, 465-474.

[11]. Bijender Singh and Shishir Jain, (2005). Semi-compatibility, compatibility and fixed point theorems in fuzzy metric space. *Journal of the Chungcheong Mathematical Society* **18**(1), 1-23.

[12]. M.R. Singh and Y.R. Singh, (2007). Compatible mappings of type (E) and common fixed point theorems of Meir-Keeler type. *Internat. J. Math. Sci. & Engg. Appl.,* **12**(2007), 299-315.

[13].V Srinivas, B.V.B. Reddy and R.U. Rao, (2013). A common fixed point theorem using A-Compatible and S-compatible mappings. *International Journal of Theoretical & Applied sciences*, **5**(1): 154-161.

[14]. Abdul Latif and Fawzia Y. Shaddad (2010). Fixed point results for multivalued maps in cone metric spaces*, Fixed Point Theory and Applications* no. 2, 11.
[15]. I. Altun, B. Damjanovi, Dragan Djoric, (2010). Fixed point and common fixed point theorems on ordered cone metric spaces. *Appl. Math. Lett*. **23**, 310-331.
[16]. Song, G., Sun, X., Zhao, Y., & Wang, G. (2010). New common fixed point theorems for maps on cone metric spaces. *Applied Mathematics Letters*, **23**(9), 1033-1037.
[17]. N. Metiya and B.S. Choudhury, (2010). Fixed points of weak contractions in cone metric spaces, *Nonlinear Analysis,* **72**, 1589-1593.
[18]. R.P. Pant, Rakesh Mohan and P.K. Mishra, (2011). Some common fixed point theorems in cone metric spaces. *IJSTM,* Vol. **2**, Issue 2, 48-56.